Quaternion group based quantization of SL(4,R)

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Abstract

Dr R. A. Wilson proposes the finite–group quantisation of the real form $SL(4,\mathbb{R})$ electroweak sector should be quantised by a specific finite subgroup of $SO(4)$, built from two quaternion groups Q_8, together with distinct left– and right–handed triplet extensions. His construction yields a group of order 288, with a normal $Q_8 \times Q_8$ subgroup and quotient $S_3 \times \mathbb{Z}_3$, and is intended to encode parity violation, flavour structure, and the observed asymmetry between $SU(2)_L$ and $SU(2)_R$.

The continuous real–form chain

$$SL(4,\mathbb{R}) \supset SO(3,3) \supset SO(4) \simeq (SU(2)_L \times SU(2)_R)/\mathbb{Z}_2,$$

arises in twistor–based unification. Using the quaternionic model of $SO(4)$, we construct explicit embeddings of $Q_8^{(L)}$ and $Q_8^{(R)}$ into $SU(2)_L$ and $SU(2)_R$, adjoin Wilson's broken and unbroken triplet symmetries, and obtain a canonical finite subgroup

$$H_{288} \subset SU(2)_L \times SU(2)_R \subset SO(4) \subset SL(4,\mathbb{R}),$$

realising his proposed quantisation at the electroweak stage. We also show how the non–commuting triplet structure of H_{288} aligns with the continuous strong–sector chain $SL(3,\mathbb{R}) \to SU(3)_{\text{eff}}$, in which colour is emergent rather than fundamental. Finally, we indicate how this finite subgroup fits into the broader twistor symmetry $SL(4,\mathbb{C})$, providing a discrete analogue of the $SL(4,\mathbb{R})$ branch complementary to the compact $SU(4)$ and conformal $SU(2,2)$ real forms.

1 SO(4) as $(SU(2)_L \times SU(2)_R)/\mathbb{Z}_2$ and quaternion actions

We recall the standard identification

$$SO(4) \;\simeq\; \frac{SU(2)_L \times SU(2)_R}{\mathbb{Z}_2}, \tag{1}$$

where $SU(2)_L$ and $SU(2)_R$ act on the quaternion space $\mathbb{H} \simeq \mathbb{R}^4$ by

$$(q_L, q_R) : \; x \;\mapsto\; q_L \, x \, q_R^{-1}, \qquad x \in \mathbb{H}, \tag{2}$$

with $q_L, q_R \in SU(2) \simeq \{q \in \mathbb{H} : |q| = 1\}$.

Writing $\mathbb{H}$ in the basis $\{1, i, j, k\}$, we identify

$$\mathbb{H} \;\simeq\; \mathbb{R}^4, \qquad x = x_0 + x_1 i + x_2 j + x_3 k \;\leftrightarrow\; (x_0, x_1, x_2, x_3)^T. \tag{3}$$

Left and right multiplication by unit quaternions then define real 4×4 matrices, giving an explicit embedding

$$SU(2)_L \times SU(2)_R \;\hookrightarrow\; SO(4) \;\hookrightarrow\; SL(4, \mathbb{R}). \tag{4}$$

For example, left multiplication by i acts as

$$L_i(x) = ix, \tag{5}$$

which in the $(1, i, j, k)$ basis is represented by

$$L_i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{6}$$

and similarly for L_j, L_k and the right actions R_i, R_j, R_k.

2 Two copies of Q_8 inside $SU(2)_L \times SU(2)_R$

The quaternion group

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} \subset SU(2) \tag{7}$$

has the usual relations

$$i^2 = j^2 = k^2 = ijk = -1. \tag{8}$$

We define two copies:

$$Q_8^{(L)} = \{\pm 1_L, \pm i_L, \pm j_L, \pm k_L\} \subset SU(2)_L, \tag{9}$$

$$Q_8^{(R)} = \{\pm 1_R, \pm i_R, \pm j_R, \pm k_R\} \subset SU(2)_R. \tag{10}$$

Then

$$Q_8^{(L)} \times Q_8^{(R)} \ \subset \ SU(2)_L \times SU(2)_R \ \subset \ SO(4) \ \subset \ SL(4,\mathbb{R}). \tag{11}$$

This 16-element subgroup is the natural finite "core" of the $SU(2)_L \times SU(2)_R$ electroweak stage in the $SL(4,R)$ real-form chain.

3 Triplet extensions: binary tetrahedral vs broken triplet

We now adjoin triplet symmetries separately to $Q_8^{(L)}$ and $Q_8^{(R)}$.

3.1 Unbroken triplet symmetry in $SU(2)_R$

The binary tetrahedral group $2T \subset SU(2)$ is a 24-element extension of Q_8 obtained by adjoining

$$t = \frac{-1 + i + j + k}{2}, \tag{12}$$

and its conjugates. We take

$$2T_R = \langle Q_8^{(R)}, t_R \rangle \subset SU(2)_R, \tag{13}$$

with t_R the right-handed analogue of t. This gives an unbroken triplet symmetry on the right-handed side.

3.2 Broken triplet symmetry in $SU(2)_L$

On the left-handed side we pick i_L as special and adjoin

$$s_L = \frac{-1 + i_L\sqrt{3}}{2}, \tag{14}$$

which generates a different 24-element extension

$$G_L = \langle Q_8^{(L)}, s_L \rangle \subset SU(2)_L, \tag{15}$$

with a distinguished i_L direction corresponding to the third component of weak isospin.

3.3 Product group and order 288 subgroup

The direct product

$$H = G_L \times 2T_R \subset SU(2)_L \times SU(2)_R \tag{16}$$

has order

$$|H| = 24 \times 24 = 576. \tag{17}$$

Quotienting by a central $\mathbb{Z}_2$ (identifying $(-1_L, -1_R)$ with $(1_L, 1_R)$) gives a subgroup of order

$$|H/\mathbb{Z}_2| = 288, \tag{18}$$

which matches the finite group structure described by Wilson: a normal subgroup of order 16 (the $Q_8^{(L)} \times Q_8^{(R)}$ core) and a quotient of order 18 ($S_3 \times \mathbb{Z}_3$ arising from the triplet symmetries).

Thus we obtain an explicit finite subgroup

$$H_{288} \subset SU(2)_L \times SU(2)_R \subset SO(4) \subset SL(4, \mathbb{R}), \tag{19}$$

realising the finite quantisation of the $SU(2)_L \times SU(2)_R$ stage in the $SL(4, R)$ real-form chain.

4 Embedding into the $SL(4, R)$ real-form chain

In the twistor unification framework we have the real-form chain

$$SL(4, R) = \mathrm{Spin}(3, 3) \supset SO(3, 3) \supset SO(4) \simeq \frac{SU(2)_L \times SU(2)_R}{\mathbb{Z}_2}, \tag{20}$$

with the electroweak decomposition

$$SO(3,3) \;>\; SO(4) = SU(2)_L \times SU(2)_R \;\to\; SU(2)_L \times U(1)_{EM}. \tag{21}$$

The finite subgroup H_{288} sits precisely at the $SU(2)_L \times SU(2)_R$ level, providing a discrete quantisation of the left/right electroweak symmetry within the $SL(4,R)$ branch.

5 Colour vs flavour and $SL(3,R) \to SU(3)_{\text{eff}}$

In the strong sector, neutrino–antineutrino composite shells naturally lead to the split real form

$$SL(3,R), \tag{22}$$

with Lie algebra

$$\mathfrak{sl}(3,R) = \{A \in M_3(\mathbb{R}) \mid \mathrm{tr}(A) = 0\}. \tag{23}$$

Compactification of this split form yields an effective colour group

$$SL(3,R) \;\to\; SU(3)_{\text{eff}}, \tag{24}$$

with gluons represented as composite excitations of the neutrino–antineutrino vacuum shells.

In this picture, non-commuting triplet symmetries in the finite group H_{288} are naturally modelled, at the continuous level, by $SL(3,R)$ rather than by a commuting torus in $U(3)$. The emergent $SU(3)_{\text{eff}}$ is a compactification of $SL(3,R)$, not a fundamental gauge group, consistent with the view that colour is not an observable symmetry but an effective description of strong interactions.

6 Relation to $SL(4,C)$ and twistor real forms

The global complex symmetry

$$SL(4,C) = \mathrm{Spin}(6,C) \tag{25}$$

admits three real forms relevant to four-dimensional physics:

$$SU(4), \qquad SL(4,R), \qquad SU(2,2), \tag{26}$$

with the twistor chain

$$SU(2,2) \subset SL(4,C) \supset SL(4,R), SU(4). \tag{27}$$

The finite subgroup $H_{288} \subset SL(4,R)$ can be viewed as a discrete twistor-like structure for the $SL(4,R)$ branch, complementing the continuous twistor geometry of the $SU(2,2)$ real form and the compact internal symmetry of the $SU(4)$ real form. Table 1 summarizes Dr Wilson's own analysis of the physics of his quantization method.

Uniqueness of the finite subgroup H_{288} under discrete constraints

We impose the following physically motivated constraints on finite subgroups of the electroweak real–form chain

$$SL(4,R) \supset SO(3,3) \supset SO(4) \simeq (SU(2)_L \times SU(2)_R)/\mathbb{Z}_2 :$$

- **(C1) Finite subgroups of $SU(2)_L$ and $SU(2)_R$:** only the standard non–abelian finite subgroups of $SU(2)$ are allowed: Q_8, $2T$, $2O$, $2I$.
- **(C2) Presence of Q_8 and Z_3:** Q_8 (discrete spin) and Z_3 (three flavours) must both appear.
- **(C3) Chirality:** $SU(2)_L$ and $SU(2)_R$ must have *different* finite subgroups.
- **(C4) Parity violation:** the left–handed subgroup must break a symmetry that the right–handed subgroup preserves.
- **(C5) Flavour directionality / mass hierarchy:** the finite group must distinguish directions in flavour space.
- **(C6) Clean embedding into $SO(4) \subset SL(4,R)$:** the finite subgroup must sit naturally inside $SU(2)_L \times SU(2)_R \subset SO(4) \subset SL(4,R)$.

Under these constraints, the relevant ways to combine Q_8 and Z_3 are: the direct product $Z_3 \times Q_8$, and the two nontrivial semidirect products $Z_3 \rtimes Q_8$

and $Q_8 \rtimes Z_3$. Up to isomorphism, these exhaust the nontrivial 24–element combinations of Z_3 and Q_8 that embed in $SU(2)$.

We now compare the main possibilities in Table 2 (on the last page of this paper).

In particular, once one insists on a finite subgroup of $SU(2)_L \times SU(2)_R$ containing both Q_8 and Z_3, with different behaviour of left– and right–handed sectors, and a minimal mechanism for weak chirality and flavour directionality, the asymmetric choice

$$SU(2)_L : \ Z_3 \rtimes Q_8, \qquad SU(2)_R : \ Q_8 \rtimes Z_3,$$

with core $Q_8^{(L)} \times Q_8^{(R)}$ and quotient $S_3 \times Z_3$, is essentially the unique minimal solution. This is precisely the finite subgroup

$$H_{288} = (G_L \times 2T_R)/\mathbb{Z}_2 \subset SU(2)_L \times SU(2)_R \subset SO(4) \subset SL(4,R),$$

constructed in this paper.

References

Wilson, R. A, Quantisation of SL(4,R), August 2, 2026. https://robwilson1.wordpress.com/2
of-sl4r/

Cook, N. B., Unification by SL(4,C) Twistors, July 31, 2026 https://ai.vixra.org/pdf/2608.00

Group	Description	Physical interpretation
Q_8	Quaternion group (order 8). Required in quantum mechanics: non-commutativity and elements squaring to -1. Represents the minimal discrete spin structure.	Finite skeleton of $SU(2)_L$ and $SU(2)_R$. Forms the 16-element core $Q_8^{(L)} \times Q_8^{(R)} \subset SO(4) \subset SL(4,R)$, encoding chiral spin and parity structure.
Z_3	Cyclic group of order 3. Minimal mathematical structure for three flavours of electrons (e, μ, τ) and, more generally, three fermion generations.	Discrete flavour triplet underlying the strong-sector chain $SL(3,R) \to SU(3)_{\text{eff}}$, where colour and flavour are emergent rather than fundamental.
$Z_3 \rtimes Q_8$	Semidirect product where Z_3 acts on Q_8, breaking the symmetry of Q_8 and coupling the flavour triplet to the quaternionic spin structure.	Left-handed weak interaction symmetry. Realised as the broken triplet $G_L \subset SU(2)_L$. Produces weak-isospin asymmetry and parity violation, with a distinguished left-handed direction corresponding to T_3.
$Q_8 \rtimes Z_3$	Semidirect product where Q_8 acts on Z_3, breaking the symmetry of Z_3 and putting a direction on the flavour triplet.	Right-handed flavour symmetry. Realised as the unbroken triplet $2T_R \subset SU(2)_R$. Gives three right-handed flavour directions and a mass hierarchy, by orienting the triplet of generations.
S_3	Permutation group of three objects, acting on the flavour triplet by permuting the three generations.	Discrete analogue of CKM-like mixing: flavour permutations appear as part of the quotient $H_{288}/(Q_8^{(L)} \times Q_8^{(R)})$, encoding generation mixing in the finite-group framework.
$\mathbb{Z}_3$ (phase)	Cyclic group of order 3 arising as a phase symmetry on the flavour triangle (clockwise vs anti-clockwise rotations).	Provides flavour directionality and contributes to mass hierarchy. The same $\mathbb{Z}_3$ structure appears in the compactification $SL(3,R) \to SU(3)_{\text{eff}}$, linking finite flavour phases to emergent colour.
H_{288}	Combined structure $(G_L \times 2T_R)/\mathbb{Z}_2$, a finite subgroup of $SU(2)_L \times SU(2)_R \subset SO(4) \subset SL(4,R)$, with normal core $Q_8^{(L)} \times Q_8^{(R)}$ and quotient $S_3 \times \mathbb{Z}_3$.	Full discrete electroweak + flavour + proto-colour symmetry in the $SL(4,R)$ branch. Provides a discrete twistor-like structure: all allowed discrete chirality flips, weak-isospin asymmetries, flavour permutations, mass-hierarchy directions, and proto-colour phases acting on an $SL(4,R)$ twistor-spinor.

Choice	**Properties**	**Failure mode**
Symmetric choice: same finite group in $SU(2)_L$ and $SU(2)_R$ (e.g. $2T_L = 2T_R$, $2I_L = 2I_R$)	Satisfies (C1), (C2). Rich finite structure; can encode flavour and colour symmetries.	Violates (C3), (C4): no chirality, no parity violation. Left and right behave identically. Physically equivalent to traditional $F_4/E_6/E_8$–type models with symmetric $SU(2)_L/SU(2)_R$.
Direct product $Z_3 \times Q_8$ in one or both $SU(2)$ factors	Satisfies (C1), (C2). Gives discrete spin and three flavours. Embeds in $U(2)$ in the usual way.	No mixing between Z_3 and Q_8: flavour and spin are decoupled. Fails (C4), (C5): no parity violation, no flavour directionality or mass hierarchy from the finite group. Phenomenologically ruled out.
Larger binary groups without restriction (e.g. full $2I$ in $SU(2)_L$, full $2T$ in $SU(2)_R$)	Satisfies (C1), (C2). Very rich finite structure; embeds in $H_4 \subset E_8$.	Overkill for the $SL(4,R)$ electroweak stage. Fails (C5), (C6): too many symmetries, no minimal mechanism for chirality, flavour directionality and clean $SO(4) \subset SL(4,R)$ embedding without ad hoc constraints.
Wilson's choice: $Z_3 \rtimes Q_8$ in $SU(2)_L$, $Q_8 \rtimes Z_3$ in $SU(2)_R$	$Z_3 \rtimes Q_8$ (dicyclic) breaks Q_8 symmetry on the left: weak–isospin asymmetry and parity violation (C3), (C4). $Q_8 \rtimes Z_3$ (binary tetrahedral) breaks Z_3 symmetry on the right: flavour directionality and mass hierarchy (C5). Both embed naturally in $SU(2)_L \times SU(2)_R \subset SO(4) \subset SL(4,R)$ (C6).	Satisfies all constraints (C1)–(C6). No obvious failure mode under the stated assumptions.

Table 2: Comparison of finite-group choices under the constraints (C1)–(C6). Only Wilson's asymmetric dicyclic/binary-tetrahedral split simultaneously realises chirality, parity violation, flavour directionality and a clean embedding in the $SL(4,R)$ twistor real-form chain.